Analysis of Expander Network on the Hypercube

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ABSTRACT

One key obstacle which has been identified in achieving parallel processing is to communicate effectively between processors during execution. One approach to achieving an optimal delay time is to use expander graph. The networks and algorithms which are based on expander graphs are successfully exploited to yield fast parallel algorithms and efficient design. The AKS sorting algorithm in time $O(\log N)$ which is an important result is based on the use of expanders. The expander graph also can be applied to construct a concentrator and a superconcentrator. Since Margulis found a way to construct an explicit linear expander graph, several expander graphs have been developed. But the proof of existence of such graphs is in fact provided by a nonconstructive argument. We investigate the expander network on the hypercube network. We prove the expansion of a single stage hypercube network and extend this from a single stage to multistage networks. The results in this paper provide a theoretical analysis of expansion in the hypercube network.

하이퍼큐브에서의 익스팬드 네트워크 분석

이 종 극†

요 약

병렬처리 과정 중에서 고려되어야할 가장 중요한 점은 프로세서사이에 통신을 어떻게 효율적으로 처리하는가 하는 것이다. 그 중 하나의 접근방법이 익스팬드 그래프를 이용하여 최적의 지연시간을 달성하는 방법이다. 익스팬드 그래프를 기초로 하여 효율적인 네트워크 구성과 수행시간이 빠른 병렬 알고리즘을 개발하기 위한 시도가 이루어져왔다. 병렬알고리즘 수행에서의 중요한 결과인 $O(\log N)$ 시간의 AKS 정렬 알고리즘은 익스팬드를 기초로 한다. 익스팬드 그래프는 다시 집중기(concentrator)와 초집중기(superconcentrator)에 적용될 수 있으며 Margulis가 선형 익스팬드 그래프의 구성하는 방법을 구체적으로 제시한 후 몇 개의 익스팬드가 제시되었다. 그러나 익스팬드 그래프를 이용한 구체적인 구조는 제시하지 않았다. 본 논문에서 hypercube 구조에서의 익스팬드 네트워크 구조를 조사하고 그리고 각 단에서의 확장성을 분석하고 다단으로 확장한다. 본 논문은 hypercube에서의 익스팬드 네크워크의 이론적 분석을 제시한다.

1. Introduction

In order to route N streams of information efficiently in parallel computer, it is necessary to construct a network with N disjoint from source to destination. Since parallel computing can employ a wide range of parallel algorithms and data structures, the most powerful interconnection scheme is one that can accommodate arbitrary source destination pairings for all N information streams. One way to build such an interconnection is to use a superconcentrator to divide the input stream into two output parts, then recursively divide each part of the output with two additional superconcentrators, and so on until each stream has been connected to its specific destination. A simple structure which can readily be applied to

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the task is the Benes network, which is difficult to route point-to-point[1], but which is easy to route as a superconcentrator[2]. The difference is that superconcentrator does not require that specific source-destination pairings be establish, only that each input be connected to one of the two output streams.

Pippenger[3] ,Valient[4] and Pinsker[5] showed how to build as superconcentrator by using the concentrator in both input and output side. Gabber and Galil[6] have introduced a family of bipartite structures call expanders, which can be used to build such concentrators. Thus, the expander graph is the key building block to construct such a concentrator, superconcentrator and routing network. Recently, Leighton and Maggs[7] showed that a randomly generated concentrator-based splitter network can be constructed with efficient routing properties and also Oruç and Guo[8] built a single stage sparse crossbar concentrator using bipartite graph.

The hypercube network is one of the most versatile and efficient networks yet discovered for parallel computation[9-11]. The focus of research is to investigate the expansion properties of the hypercube network in order to develop communication methods for parallel processor systems.

2. EXPANDER GRAPHS

The Expander graph[12] is defined by a bipartite graph. A bipartite graph $G(I, O, E)$ has a set I of inputs, a set O of outputs, and a set E of edges which connect between inputs and outputs in Fig.1 (usually consider $|I| = |O| = N$). For any subset X of input nodes, we define a subset of output nodes $\Gamma(X)$. Each node in $\Gamma(X)$ should have at least one edge which is connected to a node in subset X. If the degree of every node in the graph G is the same, then we say G is a k-regular bipartite graph. For $0 < \alpha < 1$, $\beta > 1$, and $|I| = |O| = N$, a k-regular bipartite $G(I,O,E)$

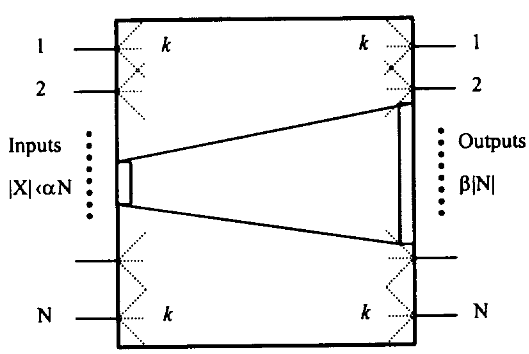

Fig. 1. (k, α, β) Expander Graph

is called a (k, α, β) expander if, for all $X \subset I$ so that $|X| \le \alpha N$, $\Gamma(X)$ in a set of output nodes is such that $|\Gamma(X)| \ge \beta|X|$. Here, β is called the expansion factor in the expander graph. In an expander graph, the number of output nodes, $\Gamma(X)$, is always larger than the number of input nodes, X.

The expanding property can be applied to interesting computation and communication properties. It also offers the means of realizing communication problems without storage capacity. Multiple connectivity between the processing elements provides the necessary redundancy for fault-tolerant communication networks. The proof of existence of such graphs is in fact provided by a nonconstructive argument. In this proof it is argued that the fraction of the graphs which do not satisfy the condition of the expander property is less than 1 when each side has N nodes with a uniform distribution with degree k. Therefore, certain graphs of degree k must exist that meet the given expansion property. Alon[13] proved this expansion property using the entropy function.

$$k > \frac{H(\alpha) + H(1-\alpha)}{H(\alpha) - (1-\alpha)H(\frac{1-\alpha}{\alpha})}$$

Where $H(\alpha) = -\alpha \log_2(\alpha) - (1-\alpha)\log_2(1-\alpha)$.

Then, with probability approaching 1, as N goes to ∞, G is a $(k, \alpha, \frac{1-\alpha}{\alpha})$ expender.

Margulis[12] found a way to construct an explicit linear. expander, but his result was deficient in that he could not give an exact value for the expansion factor. From Gabber and Galil[6], an (N, k, d) expander is a bipartite graph with N inputs and N outputs and at most kN edges, such that for subset X of inputs the subset $\Gamma(X)1$ of outputs satisfied.

$$|\Gamma(X)| \geq [1+d(1-(|X|/N)]\ |X|, \qquad (1)$$

where $\Gamma(X)$ is the set of outputs connected to X.

The expander from Gabber and Galil consists of a set of N inputs when $N=M^2$, where M is any integer, and an equal number of outputs. He found explicit construction for a family of $(N, 5, d)$ expanders with $d=(2-\sqrt{3})/4$ as well as a family of $(N, 7, d)$ expanders. The inputs are connected to the outputs by a set of seven permutations, which shift each row right or left several columns, with wrap-around. Alon, Galil and Milman[14] improved the expander factor in the same permutation of Gabber and Galil.

3. SUPERCONCENTRATOR BASED ON EXPANDER

we describe a concentrator and show how to build a superconcentrator with it. An (N, θ, k) concentrator is a two-stage connection network, with N inputs, θN outputs, at most kN links from the inputs to the outputs, having the property that, for every set of inputs X such that $|X \leq N/2$, all inputs in the set X can be one-to-one connected to the outputs. Since $\theta < 1$, this property guarantees that a stream of at most $N/2$ active inputs can be connected to the output stream along disjoint paths, while $(1-\theta)N$ of the unused inputs are disconnected from the θN outputs.

In order to construct a superconcentrator from this structure, following Pippenger[3] we build a network with N inputs and an outputs, with a direct connection from each input to a corresponding output. In order to superconcentrate a set of inputs I to a set of outputs O where $|I|=|O|$, connect any inputs in I to any output in O that happens to be linked by the direct connection. If $|I| > N/2$, then at most $N/2$ of these inputs will fail to link using the direct connection. These are then passed through an (N, θ, k) concentrator, while on the output side a mirror image structure feeds the outputs. Between these two structures, a recursion of the entire superconcentrator structure is implemented, but with θN inputs and θN outputs. This structure is illustrated in Fig. 2.

For example, applying this formula to Pippenger's $(N, 2/3, 6)$ concentrator yields $S(N) = 39N$. Because of the restriction that N must be a multiple of 6, Pippenger found a slightly higher value of $S(N) = 40N$. Excluding such minor restrictions the formula is exact. In order to build such concentrators explicitly from expanders, we now define an expander. An (N, k, d) expander, as used in the context of this paper, is a two-stage network with N inputs, N outputs, with each input connected by links to k outputs. The links are chosen in such a way that, for every set of inputs X, such that $|X| \leq N/2$, the set of outputs ΓX which are connected by links to X, is larger than $|X|$ by a factor $C > 1$. For the expanders currently studied, C is given by:

$$\frac{|\Gamma X|}{|X|} \geq \left[1+d\left(1-\frac{|X|}{N}\right)\right]=C$$

In other words, the inputs are connected to more outputs by an amount C fixed by d as used in the above formula. The hardware complexity of this structure is determined by the total number

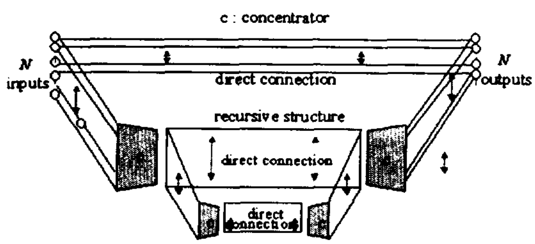

Fig. 2. A superconcentrator built with concentrators.

of links, which is kN.

The concentrator built from this expander is the union of two parts, called Part A and Part B, as shown in Fig. 3. Part A is an $(\lceil \frac{Nq}{q+1} \rceil, k, d)$ expander, Part B has $\lfloor \frac{N}{q+1} \rfloor$ inputs, with each input connected to q disjoint sets of the $\lceil \frac{Nq}{q+1} \rceil$ outputs of the expander. N' is chosen so that:

$$N \geq \lceil \frac{Nq}{q+1} \rceil + \lfloor \frac{N}{q+1} \rfloor$$

As N becomes large, $N'-N$ becomes small, so we will assume $N' \approx N$. For a specific range of values of the concentration coefficient q, $q \geq 1+2/d$, this structure is an $(N, \frac{q}{q+1}, k)$ concentrator. The concentration property is guaranteed by Hall's Matching Theorem.

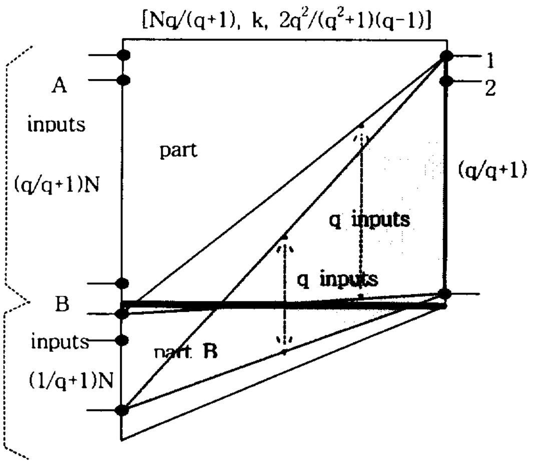

Fig. 3. The concentrator composed of two different sized parts.

4. EXPANSION ON THE HYPERCUBE

The hypercube network is one of the most versatile and efficient networks yet for parallel computation, as discussed above. The research focuses on investigating the expansion property on the hypercube network to find optimal algorithms for the hypercube. Also we demonstrate a new method of showing expansion in a network that might be applied to other networks.

This research consists of three parts. The first part is to prove the expansion property on the single stage hypercube and to find the total number of expanded outputs for given inputs. The second part is to extend a single stage to multistage. In the third part we analyze these properties of expansions and find the number of stages needed to achieve a constant expansion.

4.1 Single Stage Expansion on the Hypercube

From the bipartite graph with identity and $\log N$ connections between the input vertices I and the output vertices O where $|I| = |O|$, consider a set of binary numbered input vertices X where $|X| = x \leq N/2$ Each input vertex hart an identity edge and hypercube connections. If all the edges from X lead to Y then $|X| \leq |Y|$.

From Fig. 4 define X to be any subset of the first $N/2$ inputs, and $\Gamma_{N/2}(X)$ to be that subset of the first $N/2$ outputs which are connected to X. Define Y as any subset of the second $N/2$ inputs, and $\Gamma_{N/2}(Y)$ as that subset of the second $N/2$ outputs which are connected to Y, and let

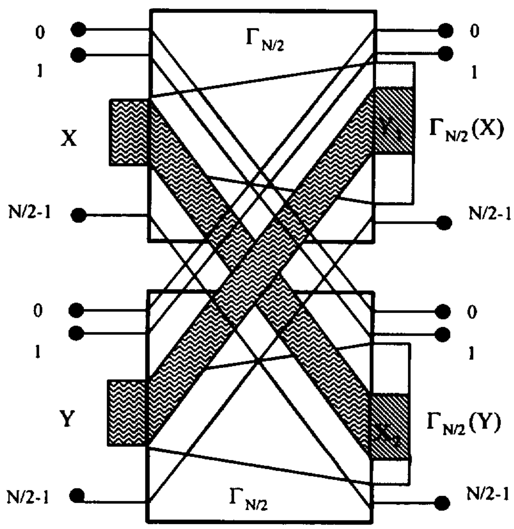

Fig. 4. An expander network built from two half size hypercube networks.

$x=|X|$, $y=|y|$, $\gamma_{N/2}(x)=|\Gamma_{N/2}(X)|$, and $\gamma_{N/2}(y)=|\Gamma_{N/2}(Y)|$ Due to symmetry of the structure the case of $x\geq y$ will be considered by choosing X accordingly. Note that X also projects an identity image X_2 onto the second half of the outputs, where $|X_2|$ = x. Similarly, Y projects an identity image Y_1 onto the first half of the outputs, where $|Y_1|$ = y.

Let us assume that for each such X, there exists Y for Which $X_2 \subseteq \Gamma_{N/2}(Y)$. Also, for this Y, assume that $Y_1 \subseteq \Gamma_{N/2}(X)$. Assume that $\gamma_{N/2}(X)$ is a convex function, and monotonically increasing for all x. Consider the case of a set of inputs Z where $Z = X \cup Y$, and $|Z| = z$. The outputs set for inputs Z is

$$\Gamma_N(Z) = \Gamma_{N/2}(X) \cup Y_1 \cup \Gamma_{N/2}(Y) \cup X_2$$

and expansion is :

$$\gamma_N(Z)=|\Gamma_{N/2}(X)\cup Y_1|+|\Gamma_{N/2}(Y)\cup X_2| \quad (2)$$

If z = x so that y = 0, then $\Gamma_N(Z)=\Gamma_{N/2}(X)\cup X_2$ and expansion is :

$$\gamma_N()=\gamma_{N/2}(X)+x \quad (3)$$

Now since γ is a monotonically increasing function, if x > y then $\gamma_{N/2}(x) \geq |Y_1|$ and for y sufficiently small $\gamma_{N/2}(y) \leq |X_2|$. Thus the smallest value of $\gamma_N(z)$ occurs ifs $Y_1 \subset \Gamma_{N/2}(X)$ so that $|\Gamma_{N/2}(X)\cup Y_1|=\gamma_{N/2}(x)$, and $\Gamma_{N/2}(Y)\subseteq X_2$ so that $|\Gamma_{N/2}(Y)\cup X_1|=x$. Thus

$$\gamma_N(z)\geq \gamma_{N/2}(x)+x$$

It follows that as x is decreased by shifting in Z from X to Y, $\gamma_N(z)$ is also decreased, until $\gamma_{N/2}(y)\geq x$.

Theorem 1 □ For an arbitrary number of input nodes z = x + y and x ≥ y, the minimum of output nodes is $\gamma_N(z)=\gamma_{N/2}(x)+\gamma_{N/2}(y)$ for x and y.

Proof) From Fig. 4, we first add a node to X recursively until $\gamma_{N/2}(y)\geq x$. Assume the first node is selected in the first half block and we always choose a second node which has the same node number with the first node among the second block.

This procedure should be done recursively until $\gamma_{N/2}(y)\geq x$. The first half block is divided into two blocks with half size and a node is selected which is the same node

number in the block among the second block. Therefore the identity images, X_2 of a set of X which is projected into the second half, should be contained by $\Gamma_{N/2}(Y)$, because X is a set of elements which is different in only one bit position in the first half. So,

$$Y_1\subseteq \Gamma_{N/2}(X) \text{ and } X_2\subseteq \Gamma_{N/2}(Y).$$

$$\Gamma_{N/2}(X)\cup Y_1=\Gamma_{N/2}(X)$$

and

$$\Gamma_{N/2}(Y)\cup X_2=\Gamma_{N/2}(Y)$$

and

$$\Gamma_N(Z)=\Gamma_{N/2}(X)\cup\Gamma_{N/2}(Y) \text{ and } \gamma_N(z) = \gamma_{N/2}(x)+\gamma_{N/2}(y)$$

If $\gamma_{N/2}(y)=x$ then we do the same procedure recursively for Y in the second half block until $X = Y$. From $X = Y$, $\Gamma_{N/2}(Y)\supseteq X_2$ and $\Gamma_{N/2}(X)\supseteq Y_1$ are still satisfied.

Therefore, the $\gamma_{N/2}(z)$ can be divided into $\gamma_{N/2}(x)$ and $\gamma_{N/2}(y)$. The $\gamma_{N/2}(x)$ and $\gamma_{N/2}(y)$ are independent of each other and divided into two blocks independently again. When N=2, $\gamma_2(1)$=2 and $\gamma_2(2)$=2 is the minimum obviously. So, $\gamma_N(z)=\gamma_{N/2}(x)+\gamma_{N/2}(y)$ should be minimum. □

Theorem 2 □ The total Nodes $N = 2^n$ and for arbitrary k between 0 and $n-1$, given the number of inputs, z, when $z=x+y$, in order to get the minimum of $\gamma_{N/2}(z)$, x and y can be chosen by the following:

- if $2\sum_{i=0}^{k-1}\binom{n-1}{i}<z\leq 2\sum_{i=0}^{k-1}\binom{n-1}{i}+\binom{n-1}{k}$

  Then, $x=z-\sum_{i=0}^{k-1}\binom{n-1}{i}$, $y=\sum_{i=0}^{k-1}\binom{n-1}{i}$ (4)

- if $2\sum_{i=0}^{k-1}\binom{n-1}{i}+\binom{n-1}{k}<z\leq 2\sum_{i=0}^{k}\binom{n-1}{i}$

  Then, $x=\sum_{i=0}^{k}\binom{n-1}{i}$, $y=z-\sum_{i=0}^{k}\binom{n-1}{i}$ (5)

Proof) Given total $N=2^n$ nodes and $x \geq y$, we know that :

$\gamma_N(z) = \gamma_{N/2}(x) + \gamma_{N/2}(y)$

We can divide into two groups. Define $group_N(k)$ and $group_N(k')$ according to z. When x is increased with fixed y in total nodes N, we say that z is in $group_N(k)$. It is in Eq.4 of the Theorem. When y is increased with fixed x in total nodes N, z is in $group_N(k')$. It is in Eq.5 of the Theorem. The $|group_N(k)|$ is the number of elements in $group_N(k)$ and $|group_N(k)| = |group_N(k')|$. If z = 1 then this is obvious, x = 1, y = 0. This will be the $group_N(0)$. If z = 2 then x = 1 and y = 1 will be one of the solutions to keep the minimum number of output nodes $\gamma_N(z) = \gamma_{N/2}(x) + \gamma_{N/2}(y)$. This is the $group_N(0')$.

$|group_N(0)| = |group_N(0')| = 1$. when y = 1, then $\gamma_{N/2}(y=1) = \log N/2 + 1$. That means we can increase the number of x in first half block until $x = \gamma_N(y=1)$ with $X_2 \subseteq \Gamma_{N/2}(y=1)$ as mentioned from Theorem 1). let's call the nodes from x = 2 to $x = \gamma_N(y=1)$ $group_N(1)$. Now, we can apply this process for y in the second half from Fig.4: that is, y can be increased until x = y instead of x without violating the previous condition to keep minimum $\gamma_N(z)$. It will be x = y = log N. This will be called $group_N(1')$ from y = 2 to y = x. The $group_N(1)$ and $group_N(1')$ have *log N/2* elements.

When $N = 2^n$, $|group_N(k)|$ is :

$$|group_N(n-1)| = \gamma_{N/2}(\gamma_{N/2}(\gamma_{N/2}\cdots(\gamma_{N/2}(1)))\cdots)_{n-1} - \gamma_{N/2}(\gamma_{N/2}(\cdots(\gamma_{N/2}(1)))\cdots)_{n-2} \quad (6)$$

where $\gamma_{N/2}(\gamma_{N/2}(\gamma_{N/2}\cdots(\gamma_{N/2}(1)))\cdots)_{n-1} = N/2$. Subscript "*n-1*" means that $\gamma_{N/2}$ is repeated n times recursively.

In case of $2N = 2^{n+1}$:

$$|group_{2N}(n)| = \gamma_N(\gamma_N(\gamma_N\cdots(\gamma_N(1)))\cdots)_n - \gamma_N(\gamma_N(\cdots(\gamma_N(1)))\cdots)_{n-1}, \quad (7)$$

For arbitrary k $(0 \leq k \leq n)$, We can get that :

$$|group_{2N}(k)| = |group_N(k-1)| + |group_N(k)| \quad (8)$$

$$|group_{2N}(n-1)| = |group_N(n-2)| + |group_N(n-1)|$$

When N = 2 there is only one group, $group_2(0)$ and $group_2(0')$ and $\gamma_2(1) = 2$ is obvious. Each half of the group has one element. When H = 4 there are two groups, $group_2(0)$, $group_2(0')$ and $group_4(1)$, $group_4(1')$. These groups have one element.

$$|group_4(0)| = |group_4(0')| = \binom{1}{0} = 1 \quad (9)$$

$$|group_4(1)| = |group_4(1')| = \binom{1}{1} = 1 \quad (10)$$

From Eq.8, Eq.9 and Eq.10,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad (11)$$

Therefore

$$y = \sum_{i=0}^{k-1} |group_N(i)| = \gamma_N(\gamma_N(\gamma_N(\cdots\gamma_N(1)))\cdots)_{k-1} = \sum_{i=0}^{k-1}\binom{n-1}{i}$$

For $group_N(k')$, we can get the same results because of $|group_N(k)| = |group_N(k')|$.

So,

$$x = \sum_{i=0}^{k-1} group_N(i) = \gamma_N(\gamma_N(\gamma_N\cdots\gamma_N(1)))\cdots)_{k-1} = \sum_{i=0}^{k-1}\binom{n-1}{i}$$

In addition, the summation of all elements in groups should be $N=2^n$.

$$\sum_{i=0}^{n-1} |groupN\ (i)| + \sum_{i=0}^{n-1} |groupN\ (i')| = 2 \cdot \sum_{i=0}^{n-1}\binom{n-1}{i} = 2^n$$

Thus, theorem 2) is proved by Eq.6 and Eq.11.□

Theorem 3 □ If z is $\sum_{i=0}^{k}\binom{n-r}{i}$ then $x = \sum_{i=0}^{k}\binom{n-r-1}{i}$ and $y = \sum_{i=0}^{k-1}\binom{n-r-1}{i}$.

That is:

$$\gamma_{N/2^r}\left(\sum_{i=0}^{k}\binom{n-r}{i}\right) = \gamma_{N/2^{r+1}}\left(\sum_{i=0}^{k}\binom{n-r-1}{i}\right) + \gamma_{N/2^{r+1}}\left(\sum_{i=0}^{k-1}\binom{n-r-1}{i}\right)$$

where $0 \le k \le n-1$ and $k \ge n-r-1$.

Proof)

$$\sum_{i=0}^{k}\binom{n-r}{i} = 2 \cdot \sum_{i=0}^{k-1}\binom{n-r}{i}+\binom{n-r-1}{k}$$

From Theorem 2), x and y should be :

$$y = \sum_{i=0}^{k-1}\binom{n-r-1}{i} \text{ and } x = \sum_{i=0}^{k}\binom{n-r}{i} - \sum_{i=0}^{k-1}\binom{n-r-1}{i}$$

and

$$x = \sum_{i=0}^{k}\binom{n-r-1}{i} + \sum_{i=0}^{k-1}\binom{n-r-1}{i} - \sum_{i=0}^{k-1}\binom{n-r-1}{i}$$

So, we can get $x = \sum_{i=0}^{k}\binom{n-r-1}{i}$ □

Thus, to find minimum $\gamma_{N/2^r}(z)=\gamma_{N/2^{r+1}}(x)+$, $\gamma_{N/2^{r+1}}(y)$, After repeating n times, this equation can be simplified by following :

$$2 \cdot \left[\sum_{j=0}^{n-1}\binom{n-1}{j}\, \gamma_{N/2^n}\Big(\sum_{i=0}^{k-j-1}\binom{n-n}{i}\Big)\right] \tag{12}$$

Here

if $\alpha>0$, then $\sum_{i=0}^{\alpha}\binom{n-n}{i}=1$, and $\gamma_{N/2^n}(1)=1$ (13)

if $\alpha=-1$, then $\sum_{i=0}^{\alpha}\binom{n-n}{i}=0$, and $\gamma_{N/2^n}(0)=1$ (14)

if $\alpha<-1$, then $\gamma_{N/2^n}(\cdot)=0$ (15)

Solve Eq.12 using Eq.13, Eq.14 and Eq.15

$$= 2 \cdot \sum_{i=0}^{k}\binom{n-1}{i} \tag{16}$$

If z is $2 \cdot \sum_{i=0}^{k-1}\binom{n-1}{i}+\binom{n-1}{k}$ then z should be divided into :

$$x=\sum_{i=0}^{k-1}\binom{n-1}{i}+\binom{n-1}{k} \text{ and } y=\sum_{o=0}^{k-1}\binom{n-1}{i}$$

From Theorem 2), after repeating n times :

$$= \sum_{j=0}^{n}\binom{n}{j} \cdot \gamma_{N/2^n}\Big(\sum_{i=0}^{k-j-1}\binom{n-n}{i}\Big) \tag{17}$$

Therefore we can get $\gamma_N(z)$ when $z=2 \cdot \sum_{j=0}^{k+1}\binom{n-1}{i}+\binom{n-1}{k}$ from Eq.17 using Eq.13, 14 and 15,

$$\gamma_N(z)=\sum_{j=0}^{k+1}\binom{n}{j}=2 \cdot \sum_{j=0}^{k}\binom{n-1}{i}+\binom{n-1}{k+1} \tag{18}$$

Let z be :

$$z=2 \cdot \sum_{i=0}^{k-1}\binom{n-1}{i}+A \tag{19}$$

A should be in $0<A\le\binom{n-1}{k}$ from Theorem 2)

$$\binom{n-1}{k}=\sum_{i=2}^{n-k-1}\binom{n-i}{k-1}$$

We can get $\gamma_N(z)$ when A is given by the following :

$$A=\binom{n-2}{k-1}+\binom{n-3}{k-1}+\binom{n-4}{k-1}+\cdots+\binom{n-p}{k-1}$$
$$= \sum_{i=2}^{p}\binom{n-p}{k-1} \tag{20}$$

where $2 \le p \le n-k+1$

The previous equation can be simplified like the following one:

$$\gamma_N(z)=2 \cdot \sum_{j=0}^{k}\binom{n-1}{j}+\binom{n-2}{k}+\binom{n-3}{k}+\cdots+\binom{n-p}{k}$$
$$=2 \cdot \sum_{j=0}^{k}\binom{n-1}{j}+\sum_{j=0}^{p-2}\binom{n-j-2}{k} \tag{21}$$

If $p=n-k+1$, Eq.21 is :

$$\gamma_N(z)=2 \cdot \sum_{j=0}^{k}\binom{n-1}{j}+\sum_{j=0}^{n-k-1}\binom{n-j-2}{k}$$
$$=2 \cdot \sum_{j=0}^{k}\binom{n-1}{j}+\binom{n-1}{k+1} \tag{22}$$

We can get the same result with Eq.18.

4.2 Multistage Expansion on the Hypercube

To extend from single stage to multistages consider two stages as shown in Fig.5 and investigate that a two stage network has the same properties that a single stage has.

Theorem 4 □ If $\Gamma_{N/2}(X)$ and $\Gamma_{N/2}(Y)$ are independent of each other, that is, $\Gamma_{N/2}(X)\supseteq Y_1$ and $\Gamma_{N/2}(Y)\supseteq X_2$ from Theorem 1), then $\Gamma_{N/2}(\Gamma_{N/2}(X))$ and $\Gamma_{N/2}(\Gamma_{N/2}(Y))$ are also independent of each other as shown in Fig. 5; that is, $\Gamma_{N/2}(\Gamma_{N/2}(X))\supseteq Y_1$ and $\Gamma_{N/2}(\Gamma_{N/2}(Y))\supseteq X_2'$ where X_2' and $Y_1'$are the identity images of $\Gamma_{N/2}(X)$ and $\Gamma_{N/2}(Y)$ onto the first half block and the second half block at the second stage.

Proof) From Theorem 1, $X \supseteq Y$ and $\Gamma_{N/2}(X)$

$\supseteq \Gamma_{N/2}(Y)$. So, $\Gamma_{N/2}(\gamma_{N/2}(X)) \supseteq \Gamma_{N/2}(Y)$

$$\Gamma_{N/2}(\gamma_{N/2}(X)) \supseteq Y_1' \tag{23}$$

where Y_1' is the identity image of $\Gamma_{N/2}(Y)$ on the first block. On the second half block,

$$\Gamma_{N/2}(Y) \supseteq X_2$$

$$\Gamma_{N/2}(\Gamma_{N/2}(Y)) \supseteq \Gamma_{N/2}(X_2) \tag{24}$$

X_2 is the identity image of X on the second block. So, we can say that :

$$\Gamma_{N/2}(X_2) = \Gamma_{N/2}(X) \tag{25}$$

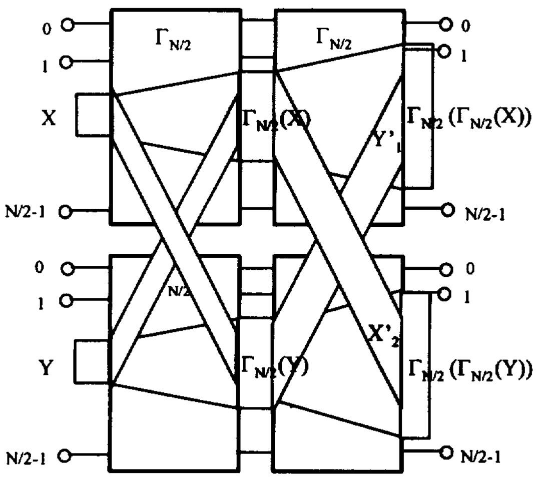

Fig. 5. A two stage expander network built from combining two hypercube

From Eq.24 and Eq.25 :

$$\Gamma_{N/2}(\Gamma_{N/2}(Y)) \supseteq \Gamma_{N/2}(X)$$

$$\Gamma_{N/2}(\Gamma_{N/2}(Y)) \supseteq X_2' \tag{26}$$

Therefore

$$\Gamma_N(\Gamma_N(Z)) = \Gamma_{N/2}(\Gamma_{N/2}(X)) \cup Y_1' \cup \Gamma_{N/2}(\Gamma_{N/2}(Y)) \cup X_2'$$

$$= |\Gamma_{N/2}(\Gamma_{N/2}(X))| + |\Gamma_{N/2}(\Gamma_{N/2}(Y))| \tag{27}$$

Theorem 5 □ Given Z inputs, X in the upper half, Y in the lower half, such that the minimum number of output connections from the first stage is $\Gamma_N(Z) = \Gamma_{N/2}(X) + \Gamma_{N/2}(Y)$ then $\Gamma_{N/2}(\Gamma_{N/2}(X)) + \Gamma_{N/2}(\Gamma_{N/2}(Y))$ is the minimum number of output connections in the second stage.

Proof) $\Gamma_{N/2}(X)$ is the minimum connection for X at the first stage outputs and $\Gamma_{N/2}(\Gamma_{N/2}(X))$ is the set of the second stage outputs for $\Gamma_{N/2}(X)$. If the input, X, is a set of input nodes to keep the minimum output nodes that are connected with X, then $\Gamma_{N/2}(Y) \supseteq X_2$ and $\Gamma_{N/2}(Y) \supseteq X$. The maximum node number of a set, X, is $\gamma_{N/2}(y)$ in the first half block according to Theorem 2. This means that $\Gamma_{N/2}(Y) = X_2$. In Fig. 4, X_2 has the same set as the set of input nodes, X. So:

$$\Gamma_{N/2}(Y) = X_2 = X \tag{28}$$

Therefore from Eq.28 and Theorem 4, $\Gamma_{N/2}(Y)$ is a set of nodes in the second half block to keep the minimum number of output nodes, $\Gamma_{N/2}(\Gamma_{N/2}(Y))$. This is the same for $\Gamma_{N/2}(X)$. □

The total minimum number of nodes which is connected to input z after two stage's expansion can be obtained like Eq.16, 18 and 21 from Theorem 4 and Theorem 5.

If the number of nodes in a set Z is given as:

$$z = 2 \cdot \sum_{i=0}^{k-1} \binom{n-1}{i} \tag{29}$$

the number of output nodes at the second stage is :

$$\gamma_N(\gamma_N(z)) = 2 \cdot \sum_{j=0}^{k-1} \binom{n-1}{j} \tag{30}$$

Let s be the number of stages and $\gamma^s{}_N(z)$ be the number of expanded outputs from any z inputs of an N node hypercube after s stages of expansion. Therefore $\gamma^s{}_N(z)$ is :

$$\gamma^s{}_N(z) = \gamma_N(\gamma_N(\cdots \gamma_N(z)))_s = 2 \cdot \sum_{i=0}^{k+s-1} \binom{n-1}{i} \tag{31}$$

When z is given as in Eq.19 and A is given as in Eq.20, $\gamma_N(\gamma_N(z))$ at the second stage is :

$$\gamma_N(\gamma_N(z)) = 2 \cdot \sum_{j=0}^{k+1} \binom{n-1}{i} + \sum_{j=0}^{k-2} \binom{n-j-2}{k} \tag{32}$$

5. ANALYSIS OF EXPANDER

When $N = 2^n$, the outputs, $\gamma_N(N/2)$ at input $z = N/2$, are very significant for expander. $\gamma_N(N/2)$ is :

When n is even

$$z=N/2=2\cdot\sum_{i=0}^{n/2-1}\binom{n}{i}$$

Therefore the total number of output nodes at N/2 inputs from the Eq.16 can be given by substituting $n/2$ for k

$$\gamma_N(N/2)=2^{n-1}+2\cdot\binom{n-1}{n/2} \tag{33}$$

Eq.33 can be written as:

$$\gamma_N(N/2)=2^{n-1}+\frac{n!}{(n/2)!(n/2)!} \tag{34}$$

When n is odd

$$z=N/2=2\cdot\sum_{i=0}^{\frac{n-1}{2}-1}\binom{n}{i}+\binom{n}{(n-1)/2}$$

Therefore the total number of output nodes at N/2 inputs from the Eq.18 can be given by substituting $(n-1)/2$ for k.

$$\gamma_N(N/2)=2^{n-1}+\binom{n}{\frac{n+1}{2}} \tag{35}$$

Total number of outputs when n is odd:

$$\gamma_N(N/2)=2^{n-1}+\frac{n!}{\frac{n+1}{2}!\cdot\frac{n+1}{2}!} \tag{36}$$

An (n,k,d) expander has n inputs and outputs and kn edges such that for every subset z of inputs:

$$\gamma_N(z)=[1+d(1-\frac{z}{N})]\cdot z$$

$$=z+d(1-\frac{z}{N})\cdot z \tag{37}$$

From this equation, the second term $d(1-z/N)\cdot z$ indicates how many more nodes can be connected to the input nodes. d is the expansion constant.

At $z=N/2$, d is:

$$\gamma_N(N/2)=[1+1/2\cdot d]\cdot z$$

$$d=2\cdot\left(\frac{\gamma_N(N/2)}{N/2}-1\right) \tag{38}$$

From Eq.34, we can derive the expansion constant, d.

$$d=\frac{n!}{2^{n-2}\cdot(n/2!)^2} \tag{39}$$

We are going to examine Eq.39 when n goes to infinity. From Eq.39, $n!$ can be approximated based on *Stirling's Formula*:

$$n!\simeq n^n\cdot e^{-n}\cdot\sqrt{2\pi n},\qquad n\to INF$$

Using *Stirling's Formula*, Eq.39 can be represented by the following:

$$d=\frac{4\cdot\sqrt{2}}{\sqrt{\pi\cdot n}} \tag{40}$$

From Eq.37, when n is odd, d is:

$$d=\frac{4\sqrt{2}}{\sqrt{\pi(n+1)}} \tag{41}$$

Or let $\beta_z=\frac{\gamma_{N(z)}}{z}$. From Eq.34, $\beta_{N/2}$ is given by approximation using *Stirling's Formula* as n grows large.

$$\beta_{N/2}=\frac{\gamma_N(N/2)}{N/2}=\frac{2\sqrt{2}}{\sqrt{\pi n}}+1 \tag{42}$$

When n is odd:

$$\beta_{N/2}=\frac{\gamma_N(N/2)}{N/2}=\frac{2\sqrt{2}}{\sqrt{\pi(n+1)}}+1 \tag{43}$$

Let us approximate the hypercube expansion into Eq.37 of Margulis when n is even $(N=2^n)$ and $\gamma_N(z)\le N/2$. We know that when

$$z=2\cdot\sum_{i=0}^{n/2-2}\binom{n-1}{i}$$

$$\gamma_N(z)\;=\;2\cdot\sum_{i=0}^{n/2-1}\binom{n-1}{i}=N/2$$

From Eq.42

$$2^{n-1}=z+d(1-\frac{z}{N})\cdot z$$

and

$$d=\frac{(2^{n-1}-z)\cdot 2^n}{(2^n-z)\cdot z} \tag{44}$$

z can be rewritten as:

$$z=2\cdot\sum_{i=0}^{n/2-2}\binom{n-1}{i}$$

$$=2^{n-1}-\frac{n!}{\frac{n}{2}!\cdot\frac{n}{2}!} \tag{45}$$

Using *Stirling's Formula*:

$$\sqrt{2\pi n}\cdot e^{-n}\cdot n^n\le n!\le\sqrt{2\pi n}\cdot e^{-n}\cdot n^n\cdot e^{1/(12n)}$$

approximate the d value into the minimum from Eq.45.

From Eq.45 the second term is:

$$\frac{n!}{\frac{n}{2}!\cdot\frac{n}{2}!}=\frac{n^n\cdot e^{-n}\cdot\sqrt{2\pi n}\cdot e^{1/(12n)}}{(n/2)^n\cdot e^{-n}\cdot\pi\cdot n\cdot e^{1/(3n)}}$$

$$= \frac{2^n \cdot \sqrt{2}}{\sqrt{\pi n} \cdot e^{1/(4n)}} \quad (46)$$

Therefore

$$z = 2^{n-1} - \frac{2^n \cdot \sqrt{2}}{\sqrt{\pi n} \cdot e^{1/(4n)}}$$

Eq.44 is:

$$d = \frac{[2^{n-1} - (2^{n-1} - \frac{2^n\sqrt{2}}{\sqrt{\pi n} e^{1/(4n)}})] \cdot 2^n}{[2^{n-1} - (2^{n-1} - \frac{2^n\sqrt{2}}{\sqrt{\pi n} e^{1/(4n)}})] \cdot (2^{n-1} - \frac{2^n\sqrt{2}}{\sqrt{\pi n} e^{1/(4n)}})}$$

$$= \frac{4 \cdot \sqrt{2\pi n} \cdot e^{1/4n}}{\pi \cdot n \cdot e^{1/4n} - 8} \quad (47)$$

Therefore

$$\gamma_n(z) \geq [1 + \frac{4 \cdot \sqrt{2\pi n} \cdot e^{1/4n}}{\pi \cdot n \cdot e^{1/(2n)} - 8} \cdot (1 - z/N)] \cdot z$$

where $z \leq \sum_{i=0}^{n/2-2} \binom{n-1}{i}$

$\gamma_N(z)$ is always greater than the total number of output nodes which is calculated by Margulis's equation using the d in Eq.47 when

$$z \leq \sum_{i=0}^{n/2-2} \binom{n-1}{i}.$$

6. CONCLUSIONS

In this research we proved that the hypercube network has expansion properties. In single stage expansion, $\gamma_N(z)$ is given by:

$$\gamma_N(z) = 2 \cdot \sum_{i=0}^{k} \binom{n-1}{i} \text{ for } z = 2 \cdot \sum_{i=0}^{k-1} \binom{n-1}{i}$$

If $z = 2 \cdot \sum_{i=0}^{k-1} \binom{n-1}{i} + \sum_{i=0}^{p} \binom{n-1}{k-1}$

$$\gamma_N(z) = 2 \cdot \sum_{i=0}^{k} \binom{n-1}{i} + \sum_{i=0}^{p-2} \binom{n-i-2}{k}$$

where $2 \leq p \leq n-k-1$.

When the input, z is $N/2$ the minimum number of output nodes, $\gamma_N(z)$ is given by $2^{n-1} + \frac{n!}{(n/2)!(n/2)!}$ for even and $2^{n-1} + \frac{n!}{((n+1)/2)!((n-1)/2)!}$ for odd were verified by computer simulations. For example when $N = 2^{100}(1.267 \times 10^{30})$ the number of the output nodes is 0.7347×10^{30}. The d value at $N/2$ is decreased as the total number of nodes increases. When N is 2^{40}, the d is around 0.5, and $d = 0.31835$ for $N = 2^{100}$. After approximating d using *Stirling's Formula* as n grows large, d is:

$$d = \frac{4\sqrt{2}}{\sqrt{\pi n}}$$

These properties of expansion will be the basis of applying a concentration or a superconcentration on the hypercube network derived from the hypercube network.

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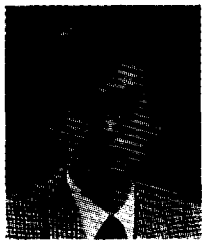

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